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"Constrained Finite Receding Horizon Linear Quadratic Control"

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Constrained Finite Receding Horizon Linear Quadratic Control

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Abstract

Issues of feasibility, stability and performance are considered for a finite horizon formulation of receding horizon control (RHC) for linear systems under mixed linear state and control constraints. It is shown that for a sufficiently long horizon, a receding horizon policy will remain feasible and result in stability, even when no end constraint is imposed. In addition, off-line finite horizon calculations can be used to determine not only a stabilizing horizon length, but guaranteed performance bounds for the receding horizon policy. These calculations are demonstrated on two examples.

Keywords: predictive control, optimal control, linear systems.

1 Introduction

Receding horizon control (RHC), also known as model predictive control (MPC), is a discrete-time technique in which the control action is obtained by repeatedly solving on-line open loop optimization problems at each time step. The flexibility of this type of implementation has been useful in addressing various implementation issues that traditionally have been problematic.

The ability to incorporate input and state constraints in optimal control design is arguably the most difficult, and yet one of the most important issues in control. Typically, constraints come in two different forms. Hard constraints (or saturation of the control input) are the most commonly encountered constraint and occur in numerous physical systems and situations due to actuator rate and amplitude saturation. So called soft constraints (constraints on the outputs of a system) on the other hand, are often imposed due to safety and/or performance considerations. In some circumstances, it may even be desirable to impose mixed constraints which relate inputs and outputs by incorporating them into a single constraint.

From a practical viewpoint, an attractive feature of RHC is its ability to naturally and explicitly handle both multivariable input and output constraints by direct incorporation into the optimization. RHC strategy was first exploited and successfully employed on linear plants, especially in the

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process industries [12, 4], where relatively slow sample times made extensive on-line intersample computation feasible.

Unfortunately, theoretical aspects associated with stability and performance properties of RHC have proven troublesome. Even when no constraints are present, stability and performance analysis of receding horizon implementations can be quite involved [7, 8, 2, 3]. The addition of end constraints (a constraint that the state be zero at the end of the output horizon) can be used to greatly simplify the analysis, but the addition of these constraints usually lacks justification or physical motivation.

When constraints are included in the problem, not only do stability and performance become more difficult, but feasibility emerges as a critical issue. An RH policy may lead to states from which the constraints are infeasible over the infinite horizon. End constraints and infinite horizon formulations may once again again be used to skirt many of these issues [10, 5].

In this paper we present a theory for stability and performance analysis of finite horizon based receding horizon control for linear systems and quadratic costs subject to mixed linear state and control constraints. Our results require no additional constraints (such as end constraints) and represent important theoretical inroads into understanding issues of feasibility, stability and performance in receding horizon implementations. This theory is balanced by presenting off–line finite horizon computational schemes for certain classes of systems which are used to provide guarantees of stability and performance for constrained RH policies. Two examples are used to demonstrate these computations.

2 Constrained Linear Quadratic Optimal Control

Consider a discrete-time linear system subject to mixed linear state and control constraints:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \quad x(0) = x_0, && (1)\\ \text{subject to:} \quad & Ex(k) + Fu(k) \le \psi && (2) \end{aligned}$$

where $x(k) \in \mathbb{R}^n$ and $u(k) \in \mathbb{R}^m$ denote the state and control, respectively. The constraints are written in vector form with $\psi \in \mathbb{R}^p$, $E \in \mathbb{R}^{p\times n}$ and $F \in \mathbb{R}^{p\times m}$. A popular design paradigm for linear time-invariant systems is linear-quadratic (LQ) optimal control [6]. The LQ optimal control problem may be posed in either an infinite or finite horizon framework.

- **Infinite Horizon Formulation**

The *infinite horizon* LQ problem is formulated as follows. Minimize the infinite horizon cost:

$$J(x_0) = \inf_{u(\cdot)} \sum_{k=0}^{\infty} \left(x^T(k)Qx(k) + u^T(k)Ru(k) \right), \tag{3}$$

subject to the system dynamics (1), and constraint (2). When it is impossible to satisfy the constraints (2) over the infinite horizon from the initial state x_0, we will resort to the convention of defining $J(x_0) = \infty$.

- **Finite Horizon Formulation**

The corresponding *finite horizon* problem is defined by the objective function:

$$J_N(x_0) = \inf_{u(\cdot)} \left[x^T(N)P_0x(N) + \sum_{k=0}^{N-1} \left(x^T(k)Qx(k) + u^T(k)Ru(k) \right) \right] \tag{4}$$

subject to the system dynamics (1), and constraint (2). Similar to the infinite horizon case, we *define* $J_N(x_0) = \infty$ when the constraints are infeasible over the horizon length N.

• Receding Horizon Formulation: Problem Setup

A receding horizon implementation is typically formulated by introducing the following open-loop optimization problem.

$$J_{(p,m)}(x_0) = \inf_{u(\cdot)} \left[x^T(p)P_0x(p) + \sum_{i=0}^{p-1} x^T(i)Qx(i) + \sum_{i=0}^{m-1} u^T(i)Ru(i) \right] \tag{5}$$

$$\text{subject to:} \quad Ex + Fu \leq \psi \tag{6}$$

($p \geq m$) where p denotes the length of the *prediction horizon*, and m denotes the length of the *control horizon*. (When $p = \infty$, we refer to this as the infinite horizon problem, and similarly, when p is finite, we refer to it as a finite horizon problem.)

Let $u^*_{(p,m)}(i)$, $i = 0, \cdots, m-1$ be the minimizing control sequence for $J_{(p,m)}(x(k))$. A receding horizon policy proceeds by implementing *only the first* control $\hat{u}_{(p,m)}(x(k)) = u^*_{(p,m)}(0)$ to obtain $x(k+1) = Ax(k) + Bu^*_{(p,m)}(0)$. The rest of the control sequence $u^*_{(p,m)}$ is discarded and $x(k+1)$ is used to update the optimization problem (5) as a new initial condition. This process is repeated, each time using only the first control action to obtain a new initial condition, then shifting the cost ahead one time step and repeating, hence the name receding horizon control.

In particular, if we consider the case $p = m = N$, then $J_{(p,m)} = J_N$ as defined in (4). This finite horizon based RH policy can then be simply characterized as:

$$\hat{u}_N(x(k)) = \arg\min_u \{x^T(k)Qx(k) + u^TRu + J_{N-1}(Ax(k) + Bu)\} \tag{7}$$

$$\text{subject to:} \quad Ex(k) + Fu \leq \psi$$

For the remainder of this paper, we assume that $p = m = N$.

3 Assumptions and Notation

We make the following assumptions concerning the constrained optimal control problem:

(i) $Q > 0$, $R > 0$, This implies $[Q^{1/2}, A]$ observable.

(ii) [A,B] controllable.

(iii) $P_0 = Q$. This implies that J_N is monotonically non-decreasing.

(iv) There exists a neighborhood of the origin which is feasible for deadbeat and the unconstrained optimal control.

Note that due to the convexity of the constraints and the cost, the infinite horizon cost $J(x)$ will also be convex and hence continuous on the set of points where J is finite.

The following notation will be used throughout:

- For any set W, let $\overset{\circ}{W}$ denote its interior, $\bar{W}$ its closure, and W^c its complement. Given two sets W and V, $W - V$ is defined as $W - V = W \cap V^c$.

- Let S_μ denote the μ sub–level set of the optimal infinite horizon cost $J(x)$ i.e.:

$$S_\mu = \{x : J(x) \leq \mu\}$$

Finally, we introduce the following definition:

Definition 3.1 *A set W is said to be* RH N–invariant *if W is an invariant set under the closed–loop system using the RH controller of horizon length N i.e. W is RH N–invariant iff*

$$x(k) \in W \Rightarrow x(k+1) = Ax(k) + B\hat{u}_N(x(k)) \in W$$

4 Feasibility and Constraints

Due to the use of a finite horizon, feasibility of the infinite horizon problem can become a serious concern in the implementation of RH policies. Finite receding horizon policies may drive the state into regions of state space from which the infinite horizon optimal control problem is unsolvable. There can be no solution in two fundamental ways. The first is that there may not even exist a feasible control and state trajectory that can satisfy the constraints over the infinite horizon. The second is that feasible control actions may exist for all time, but cannot stabilize the system, resulting in an infinite value of the infinite horizon cost.

The goal of this section is to classify the region from which the constrained optimal control problem is solvable, and then show that under an appropriately chosen horizon, RH policies always remain in this feasible region. Thus it will be demonstrated that RH policies can effectively deal with issues of feasibility without unnecessary end constraints, provided horizons are chosen properly.

4.1 I_∞: The feasible region

We begin with a characterization of the set of points from which the optimal infinite horizon cost is finite. The optimal control problem will be well defined (i.e. the infinite horizon cost will be finite) only if initial conditions in this set are considered.

The feasible region can be characterized by the following "backward" recursion. Beginning from the origin, the set of points that can reach the origin from a single step while satisfying the constraints are classified. Next, we consider the set of points that may reach the previous set in a single step. This process is carried out ad infinitum as given below:

1. Let $I_0 = \{0\}$
2. Take I_{k+1} to be

$$I_{k+1} = \{x : \exists u,\ Ex + Fu \leq \psi,\ Ax + Bu \in I_k\}$$

Define:

$$I_\infty = \bigcup_{k=0}^{\infty} I_k$$

.

Theorem 4.1 *Assume $[A, B]$ controllable, then $x \in I_\infty \Leftrightarrow J(x) < \infty$.*

Proof
Assume $x \in I_\infty$, then $\exists k$ such that $x \in I_k$. Hence, by the construction of the sets I_k, this means that there exists a sequence of k controls $\hat{u}(0), ..., \hat{u}(k)$ such that this control sequence will bring the state to zero at $x(k+1) = 0$. Hence, we have

$$J(x) \leq \sum_{i=0}^{k+1} x^T(i)Qx(i) + \hat{u}(i)^T R\hat{u}(i) < \infty$$

Now assume $x \notin I_\infty$. Since $[A, B]$ is controllable, for n equal to the size of the state and from any point in a small enough neighborhood of the origin, we may perform dead-beat control (unsaturated, Assumption (iv)), that will take the state to the origin. Hence I_n contains a neighborhood of the origin. Now, since $x \notin I_\infty$, then no sequence of controls exists such that the state will ever enter I_n. Hence for all k, the state $x(k)$ lies outside a neighborhood of the origin. Let δ be the minimum of $x^T Qx$ outside of this neighborhood. Since $\delta > 0$, we have that

$$J(x) \geq \sum_{k=0}^{\infty} \delta = \infty.$$

∎

4.2 Feasibility

While I_∞ classifies the region from which the optimal control problem has a finite solution, the question still remains as to whether a RH policy can prevent the state from leaving the feasible set I_∞. The answer to this question is provided in this section by demonstrating that the sub–level sets of $J(x)$ are RH N–invariant (Defn. 3.1).

Before presenting the feasibility theorem, we establish three preliminary results. The first states that the set of initial conditions over which any finite horizon problem is feasible is a closed set.

Lemma 4.1 *For any finite horizon length N, the constraints (2) define a closed set of initial conditions in state space.*

Proof Over the horizon length N, the constraints can be written as:

$$\begin{aligned} Ex(0) + Fu(0) &\leq \psi \\ EAx(0) + EBu(0) + Fu(1) &\leq \psi \\ &\vdots \\ EA^N x(0) + EA^{N-1}Bu(0) + \cdots + EBu(N-1) + Fu(N) &\leq \psi \end{aligned}$$

which clearly defines a closed set. ∎

The next two lemmas concern where the state can lie after a single step in a RH policy. The first shows that it must lie in a bounded set, while the second establishes that for a long enough horizon length, it must lie in the feasible set I_∞.

Definition 4.1 $W = \{x : x^T Qx \leq \mu\}$

Note that W is a compact set that contains S_μ.

Lemma 4.2 *$x(k) \in S_\mu \Rightarrow x(k+1) \in W$ under an RH policy of any horizon length N.*

Proof Note that without loss of generality, we may assume $x(k) = x(0)$. So for $x(0) \in S_\mu$, we have the following chain of inequalities:

$$\begin{aligned} \mu &\geq J(x(0)) \\ &\geq J_N(x(0)) \\ &= x^T(0)Qx(0) + \hat{u}_N^T(x(0))R\hat{u}_N(x(0)) + J_{N-1}(x(1)) \\ &\geq J_{N-1}x(1) \\ &\geq x^T(1)Qx(1) \end{aligned} \tag{8}$$

which implies that $x(1) \in W$. ∎

Lemma 4.3 *There exists a finite horizon length N such that for $x(k) \in S_\mu$, $x(k+1) \in I_\infty$.*

Proof This Lemma is a simple consequence of the arguments involved in the proof of Theorem 4.1. We recall that for $x(0) \notin I_\infty$, there exists a neighborhood of the origin which the state may never enter (cf. Theorem 4.1). Outside of this neighborhood, the minimum of x^TQx is $\delta > 0$. Then clearly for $x \notin I_\infty$, we have the following lower bound: $J_N(x) \geq N\delta$. Now, by merely choosing N such that $(N-1)\delta > \mu$, then it is clear that for $x(0) \in S_\mu$, then $x(1) \in I_\infty$, otherwise we have the contradiction:

$$\mu \geq J_N(x(0)) \geq J_{N-1}(x(1)) > (N-1)\delta > \mu$$

∎

Now we present the main feasibility theorem.

Theorem 4.2 *Let $\mu > 0$ be fixed and consider the μ sub-level set of $J(x)$, (i.e., S_μ). Then there exists an N' such that for any $N \geq N'$, S_μ is RH N-invariant.*

Proof The proof is divided into two steps. The first establishes that there exists a neighborhood of the origin from which any state in this neighborhood will not leave S_μ by the next step. By "removing" the origin in this fashion, a positive lower bound for x^TQx can be obtained for the rest of S_μ. This lower bound is used in the second portion of the proof, which relies on the compactness of the set W (Defn. 4.1) and argues by contradiction to prove the result.

Step 1 First we show that there exists a smaller sub-level set contained in S_μ, which we will denote S_β with $\beta < \mu$ such that if $x(0) \in S_\beta$, then $x(1) \in S_\mu$. In other words, if the state is in S_β, then there is no possibility of it leaving S_μ on the next step (see Figure 1).

Let β be the largest number such that

$$\{x : x^TQx \leq \beta\} \subset S_\mu$$

To show that $x(0) \in S_\beta \Rightarrow x(1) \in S_\mu$ we note that for $x(0) \in S_\beta$:

$$\begin{aligned} \beta &\geq J(x(0)) \\ &\geq J_N(x(0)) \\ &= x^T(0)Qx(0) + \hat{u}_N^T(x(0))R\hat{u}_N(x(0)) + J_{N-1}(x(1)) \\ &\geq J_{N-1}(x(1)) \\ &\geq x^T(1)Qx(1) \end{aligned}$$

From the definition of β this implies $x(1) \in S_\mu$.

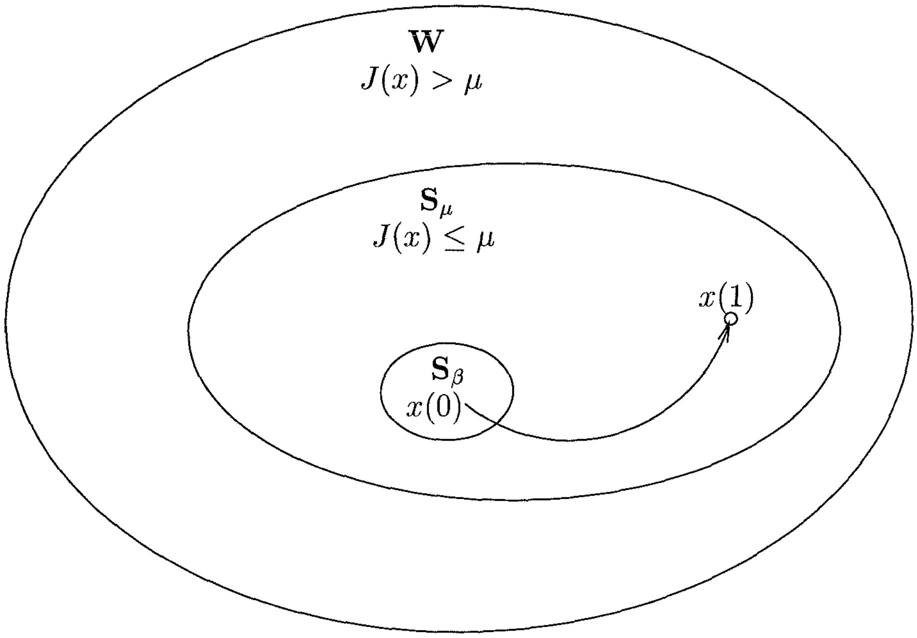

Figure 1: Step 1 of feasibility proof

Step 2 We are left to consider $x(0) \in S_\mu - S_\beta$ and prove that a sufficiently long horizon length will render S_μ RH N-invariant. The proof of this proceeds by contradiction.

Assume that for every n, there exists a horizon length $N_n \geq n$ and an $x_n(0) \in S_\mu - S_\beta$ such that $x_n(1) = Ax_n(0) + B\hat{u}_{N_n}(x_n(0)) \notin S_\mu$ (i.e. $x_n(1) \in W - S_\mu$). For notational convenience, call $x_n = x_n(1)$. Then x_n is a sequence in the compact set W. Therefore, x_n has a convergent subsequence $x_k \to x_\infty$ (See Figure 2).

Note the following properties of x_∞.

- $J(x_\infty) \geq \mu$: Since either $J(x_\infty) = \lim J(x_k)$ or $x_\infty \notin I_\infty$ so $J(x_\infty) = \infty$ (Recall from Lemma 4.3 that for some $k > K$ large enough, all the x_k lie in I_∞ and hence have well defined infinite horizon costs $J(x_k)$).
- There exists an N such that $\infty > J_N(x_\infty) > \mu - \frac{\epsilon}{4}$. For assume it is not true. Then either

    1. $J_N(x_\infty) \leq \mu - \frac{\epsilon}{4}$ for all N, in which case $J(x_\infty) \leq \mu - \frac{\epsilon}{4}$ which is not possible, or

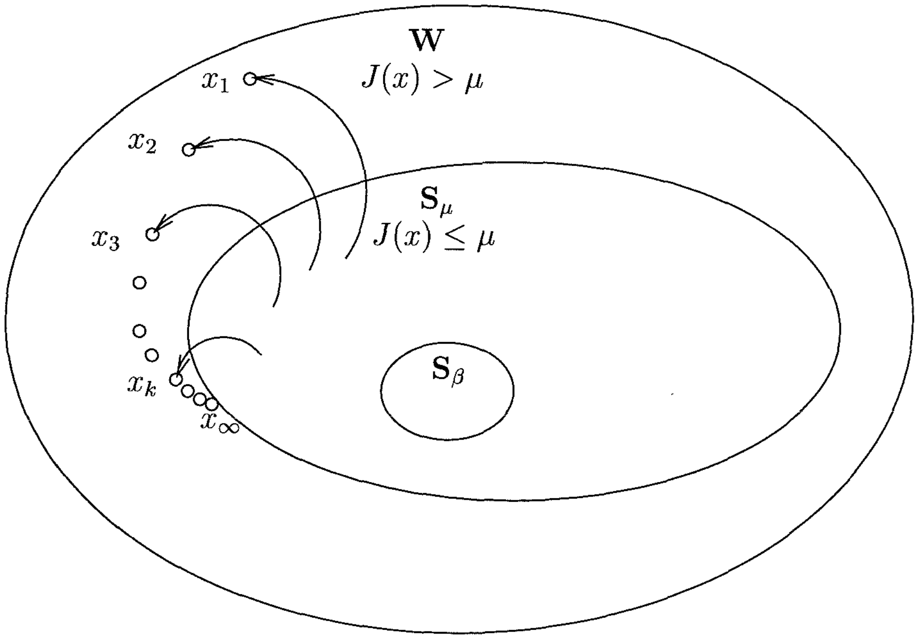

Figure 2: Step 2 of feasibility proof

2. There exists a finite N where x_∞ is not feasible for J_N. Then by Lemma 4.1 which says that the constraints define a closed set of initial conditions over any finite horizon N, we have x_∞ is not in this closed set. Hence there exists an open neighborhood of x_∞ which is not feasible for horizon lengths greater than or equal to N. This implies that $x_k \not\to x_\infty$ which is also a contradiction.

In particular, if we take ϵ to be

$$\epsilon = \min_{x \in S_\mu - \overset{\circ}{S}_\beta} x^T Q x > 0$$

then there exists an N such that

$$\infty > J_N(x_\infty) \geq \mu - \frac{\epsilon}{4}$$

By the continuity of J_N, we can choose k large enough so that $N_k > N$ and

$$J_{N_k-1}(x_k) \geq J_N(x_k) \geq \mu - \frac{\epsilon}{2}$$

Recalling that $x_k = x_k(1) = Ax_k(0) + B\hat{u}_{N_k}(x_k(0))$, and that $x_k(0) \in S_\mu$, we have the following:

$$\begin{aligned}
\mu &\geq J_{N_k}(x_k(0)) \\
&= x_k^T(0) Q x_k(0) + \hat{u}_{N_k}^T R \hat{u}_{N_k}^T + J_{N_k-1}(x_k(1)) \\
&\geq \epsilon + \mu - \frac{\epsilon}{2} \\
&= \mu + \frac{\epsilon}{2}
\end{aligned}$$

which is a contradiction. This proves the theorem. ∎

Apart from being an important result in its own right, the previous theorem is a critical step toward the goal of establishing the stability of RH policies, which will be explored in the next section.

5 Stability and Performance

In this section, the main result concerning the stability of constrained finite RH LQ policies is proven. Furthermore, it is shown that this naturally leads to a derivation of performance bounds which determine the amount by which the RH policy exceed the optimal infinite horizon cost. Before presenting these theorems we need to establish and recall some preliminary results.

5.1 Preliminaries

For a brief review of some of the concepts from real analysis that will be used here, see Appendix A. Recall the following theorem from real analysis [13]. It will play a key role in the proof of the main stability result:

Theorem 5.1 *(Dini): Let φ_N be a sequence of upper semi-continuous real-valued functions on a compact space X, and suppose that for each $x \in X$ the sequence $\varphi_N(x)$ decreases monotonically to zero. Then φ_N converges to zero uniformly.*

Proof Appendix A ∎

It is also necessary to establish the following lemma related to the continuity and convergence properties of the finite and infinite horizon costs.

Lemma 5.1 *Let I be a subset of I_∞ (c.f. Section 4.1). Define $\varphi_N(x) = \frac{J(x)-J_N(x)}{x^TQx}$, $\forall x \neq 0 \in I$, and $\varphi_N(0) = \overline{\lim}_{x\to 0}\varphi_N(x)$. Then φ_N is a sequence of upper semi-continuous functions on S_μ that converge pointwise and monotonically to zero.*

Proof Each φ_N is upper-semi continuous by definition. To prove that $\overline{\lim}_{x\to 0}\varphi_N(x) = 0$. By assumption (c.f. (iv) section 3), in a small enough neighborhood of the origin the constraints will not be active. Hence $J(x) = x^TPx$, and $J_N(x) = x^TP_Nx$ where P is the solution to the algebraic Riccati equation and P_N solves the Riccati difference equation [1, 6]. So

$$\varphi_N(x) = \frac{x^T(P-P_N)x}{x^TQx} \leq \frac{\overline{\lambda}(P-P_N)\|x\|^2}{\underline{\lambda}(Q)\|x\|^2} = \frac{\overline{\lambda}(P-P_N)}{\underline{\lambda}(Q)} \to 0$$

Since $P_N \to P$ from standard LQ theory. ∎

We also define $\varphi_0 = \frac{J(x)}{x^TQx}$ for $x \neq 0$, and $\varphi_0(0) = \overline{\lim}_{x\to 0}\varphi_0(x)$. Clearly, this function is also upper semi–continuous on S_μ.

We are now ready to prove the main stability result.

5.2 Stability

The following theorem states that given a compact set of initial conditions, there exists a single horizon length which will stabilize every initial condition using a finite RH policy.

Theorem 5.2 *Let I be a compact subset of I_∞. Then there exists an N^* such that for $N \geq N^*$, the receding horizon policy is stabilizing for any initial condition in I, and J_{N^*} serves as a Lyapunov function.*

Proof Define $\mu = \max_{x \in I} J(x)$. Once again let S_μ denote the μ sub–level set of $J(x)$. We only consider $N \geq N'$ where N' is as given in Theorem 4.2. Hence, we know that S_μ is RH n–invariant.

The idea is to use J_N as a Lyapunov function. We have the following relationship where $u(k) = \hat{u}_N(x(k))$ is the RHC:

$$J_N(x(k)) - J_N(x(k+1)) = x^T(k)Qx(k) + u^T(k)Ru(k) + J_{N+1}(x(k+1)) - J_N(x(k+1)) \qquad (9)$$

If the right hand side of (9) can be made positive, then $J_N(x(k)) > J_N(x(k+1))$ and J_N is a Lyapunov function.

Consider the functions φ_N as defined in Lemma 5.1. These functions satisfy the assumptions of Dini's Theorem on the compact set S_μ. Hence, the φ_N converge uniformly to zero on S_μ.

This implies that for every $\epsilon > 0$, there exists an N^* such that for $N \geq N^*$:

$$|\varphi_N(x)| < \epsilon$$

writing this out more explicitly gives:

$$\frac{J(x) - J_N(x)}{x^T Q x} < \epsilon$$

or

$$J(x) - J_N(x) < \epsilon x^T Q x$$

Now we also know that

$$J_{N+1}(x) - J_N(x) \leq J(x) - J_N(x) < \epsilon x^T Q x < \epsilon J_N(x)$$

This means that for any $N \geq N^*$, we have:

$$J_{N+1}(x) < (1+\epsilon) J_N(x) \qquad (10)$$

Since the function $\varphi_0(x)$ is an upper semi–continuous function on a compact set S_μ, then it achieves its maximum. For notational purposes, denote this maximum by ρ^{-1}, i.e.: For any N, we have that

$$\frac{J_N(x)}{x^T Q x} \leq \frac{J(x)}{x^T Q x} \leq \rho^{-1}$$

which implies the following inequality:

$$\rho J_N(x) \leq \rho J(x) \leq x^T Q x \qquad (11)$$

Now that we have established both (10) and (11), we are able to prove that for N large enough, J_N is a Lyapunov function. Let $u(k)$ be the RH policy, then:

$$J_N(x(k)) - J_N(x(k+1)) \quad = \quad x^T(k)Qx(k) + u^T(k)Ru(k) + J_{N-1}(x(k+1)) - J_N(x(k+1))$$

$$\begin{aligned}
&\geq x^T(k)Qx(k) + u^T(k)Ru(k) + \frac{1}{1+\epsilon}J_N(x(k+1)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad - J_N(x(k+1)) \\
&= x^T(k)Qx(k) + u^T(k)Ru(k) - \frac{\epsilon}{1+\epsilon}J_N(x(k+1)) \\
&\geq x^T(k)Qx(k) - \frac{\epsilon}{1+\epsilon}J_N(x(k+1)) \\
&\geq \rho J_N(x) - \frac{\epsilon}{1+\epsilon}J_N(x(k+1))
\end{aligned}$$

Rearranging terms gives:

$$(1+\epsilon)(1-\rho)J_N(x(k)) \geq J_N(x(k+1)) \tag{12}$$

Now since we can make ϵ as small as we like, and $0 < \rho \leq 1$, choose N^* large enough so that:

$$(1+\epsilon)(1-\rho) < 1$$

This shows that $J_{N^*}(x)$ is a Lyapunov function. ∎

To develop computational schemes later, it will be useful to use parameters that do not rely on the optimal infinite horizon cost since in general it is unknown and uncomputable. Motivated by this, we define the following quantities and follow a presentation similar to that in [9].

Define

$$\alpha_N = \min\{\alpha : \alpha J_N(x) \geq J_{N+1}(x),\ \forall x \in S_\mu\}.$$

For N larger than N^* in Theorem 5.2, we note that

$$\alpha_N \leq (1+\epsilon),$$

and that $\alpha_N \to 1$ as $N \to \infty$.

We also define

$$\rho_N = \max\left\{\rho : x^T Q x \geq \rho J_N(x),\ \forall x \in S_\mu\right\}.$$

In general we have $\rho_N \geq \rho$, where ρ is as in Theorem 5.2.

A sufficient condition for stability can be stated as:

Theorem 5.3 *Let $N \geq N'$ in Thm. 4.2 be such that*

$$\gamma_N = \alpha_{N-1}(1-\rho_N) < 1$$

then the receding horizon policy $\hat{u}_N(\cdot)$ is stabilizing, and $J_N(\cdot)$ is a Lyapunov function for the closed-loop system with

$$J_N(x(k+1)) \leq \gamma_N J_N(x(k)), \qquad \forall x(k) \in S_\mu$$

5.3 Performance

The above results lead easily to the following performance result which parallels those in [11] and [9].

Theorem 5.4 *Let N and γ_N be as in Theorem 5.3. Denote the infinite horizon performance using the receding horizon policy $u(k) = \hat{u}_N(x(k))$ by:*

$$J_{\hat{u}_N}(x(0)) = \sum_{k=0}^{\infty} x^T(k)Qx(k) + u^T(k)Ru(k) \tag{13}$$

Then a bound for the infinite-horizon performance is given by:

$$J_N(x(0)) \leq J(x(0)) \leq J_{\hat{u}_N}(x(0)) \leq \mathcal{P}_N J_N(x(0)). \tag{14}$$

where

$$\mathcal{P}_N = \left(1 + \left(\frac{\alpha_{N-1} - 1}{\alpha_{N-1}}\right)\frac{\gamma_N}{1-\gamma_N}\right) \tag{15}$$

Proof: Bounding the cost term-by-term gives the following. First,

$$\begin{aligned} x^T(0)Qx(0) + u^T(0)Ru(0) &= J_N(x(0)) - J_{N-1}(x(1)) \\ &= J_N(x(0)) - J_N(x(1)) + J_N(x(1)) - J_{N-1}(x(1)) \\ &\leq J_N(x(0)) - J_N(x(1)) + J_N(x(1)) - \frac{1}{\alpha_{N-1}}J_N(x(1)) \\ &\leq J_N(x(0)) - J_N(x(1)) + \left(\tfrac{\alpha_{N-1}-1}{\alpha_{N-1}}\right) J_N(x(1)). \end{aligned}$$

Similarly,

$$x^T(1)Qx(1) + u^T(1)Ru(1) \leq J_N(x(1)) - J_N(x(2)) + \left(\tfrac{\alpha_{N-1}-1}{\alpha_{N-1}}\right) J_N(x(2)).$$

By a summation of corresponding bounds on $x^T(k)Qx(k) + u^T(k)Ru(k)$ we obtain the following:

$$\begin{aligned} \sum_{k=0}^{\infty} x^T(k)Qx(k) + u^T(k)Ru(k) &\leq J_N(x(0)) + \left(\tfrac{\alpha_{N-1}-1}{\alpha_{N-1}}\right)\sum_{k=1}^{\infty} J_N(x(k)) \\ &\leq J_N(x(0)) + \left(\tfrac{\alpha_{N-1}-1}{\alpha_{N-1}}\right)\sum_{k=1}^{\infty} J_N(x(k)) \\ &\leq \left(1 + \left(\tfrac{\alpha_{N-1}-1}{\alpha_{N-1}}\right)\textstyle\sum_{k=1}^{\infty}\gamma_N^k\right) J_n(x(0)) \\ &= \left(1 + \left(\tfrac{\alpha_{N-1}-1}{\alpha_{N-1}}\right)\tfrac{\gamma_N}{1-\gamma_N}\right) J_N(x(0)), \end{aligned}$$

∎

Note that $\mathcal{P}_N$ bounds the amount by which the infinite horizon cost of the receding horizon policy ($J_{\hat{u}_N}$) may exceed the optimal infinite horizon cost (J). This is only true because the finite horizon costs J_N are non-decreasing as N increases. For choices of terminal weight other than $P_0 = Q$, this may not be the case, and the equivalent of the above result provides only an upper bound for the cost $J_{\hat{u}_N}$ (See Thm. B.3).

As in the linear unconstrained case [9], rather than using α_N and ρ_N to define γ_N, which effectively determines a sufficient condition for J_N to be a Lyapunov function ($\gamma_N < 1$, cf. Thm. 5.3), one may directly check whether J_N is a Lyapunov function by defining the parameter:

$$\zeta_N = \min\{\zeta : \zeta J_N(x(0)) \geq J_N(x(1)),\ \forall x(0) \in S_\mu\} \tag{16}$$

If ζ_N is less than one for $N > N'$ in Thm. 4.2, then J_N is a Lyapunov function and stability is guaranteed. Additionally, by substituting ζ for γ in Thm. 5.4, it may be used to guarantee performance as well. Note that using ζ is always less conservative than using γ. This will be clearly demonstrated in the examples.

5.4 End constraints and monotonicity

End constraints that require the state to be zero at the end of the output horizon are a common assumption in the analysis of RH policies [7, 8, 10, 5]. These constraints result in great simplifications of the above arguments.

If end constraints are feasible at $x(k)$, then feasibility is automatically guaranteed for $x(k+1)$ since the control sequence for $x(k)$ drives the state to the origin and passes through $x(k+1)$. Hence feasibility is rendered a non–issue.

Stability also becomes trivial. Following the idea in the proof of Thm 5.2 of using J_N as a Lyapunov function, the right hand side of equation (9) is trivially positive once one notes that end constraints imply that the finite horizon costs J_N are monotonically non–increasing in N. The argument is based on the fact that the optimal sequence of controls for $J_N(x(0))$ is a feasible sequence for $J_{N+1}(x(0))$ by adding a zero final control. In fact, it is immediate that J_N is a Lyapunov function for any scheme which produces monotonically non–increasing finite horizon costs. Additionally, J_N serves as an upper bound for the infinite horizon cost of the RH policy. For a more detailed explanation of the connections between end constraints, monotonicity, stability and performance bounds, see [9].

End constraints represent one extreme of the choice of possible terminal weights P_0 that can be used in RH formulations (c.f. eqn. 5 with $P_0 = \infty$). We have only considered the terminal weight $P_0 = Q$. It should be mentioned that simple modifications of the above results hold for arbitrary positive definite P_0 between Q and ∞. This extension is dealt with in Appendix B.

6 Computational Schemes

6.1 Stable plants with control saturation constraints

Consider the following stable system:

$$\dot{x} = Ax + Bu \tag{17}$$

subject to a saturation constraint on the control:

$$\|u\|_\infty \leq \psi$$

The first property we note is that for saturation constraints, the control $u = 0$ is always feasible. This means that the finite horizon costs, $J_N(x)$ are defined for all x and every N.

An additional property of stable systems that can be used to our advantage is that by using no control action, an upper bound for the constrained cost is obtained.

We outline a computational scheme below. Let I be an admissible set of initial conditions:

1. Calculate the uncontrolled cost ($u = 0$) by solving the Lyapunov equation:

   $$A^T PA - P + Q = 0$$

   The infinite horizon uncontrolled cost is given by $x^T Px$.

2. Calculate

   $$\mu = \max_{x \in I} x^T Px$$

   Define the set:

   $$W = \{x : x^T Px \leq \mu\}$$

3. Solve the optimization problems:

$$\alpha_N = \max \frac{J_{N+1}(x)}{J_N(x)}$$
$$\text{s.t. } x \in W$$

and

$$\rho_N = \min \frac{x^T Q x}{J_N(x)}$$
$$\text{s.t. } x \in W$$

or

$$\zeta_N = \max \frac{J_N(x(1))}{J_N(x(0))}$$
$$\text{s.t. } x(0) \in W$$

Note that each evaluation of $J_N(x)$ requires the solution of a quadratic program.

4. Check the condition in Thm. 5.3 (either using γ_N or ζ_N). If it is satisfied, then stability is guaranteed for every initial condition in I. (Note that this is true because the μ sublevel set of J_N is RH n-invariant and contained completely in W.)

5. Performance results from Thm 5.4 are then applicable and can be calculated.

Note that the optimization problems required to calculate α_N, ρ_N and ζ_N are in general difficult nonconvex optimization problems and no guarantee of a global optimum is possible. However, stability can only be guaranteed when a global optimum is found. Hence, the results obtained by the above scheme are only as valid as one's confidence in the solutions to the optimization problems in Step 3. With this caveat, we shall continue to use the term "guarantee" in a loose sense when referring to results obtained in the above manner.

6.2 State constraints which bound a feasible region

Computation is also possible when the constraints possess the property of bounding a feasible region of state space. Call this region W. The constraints then ensure that the trajectory remains feasible for all time (since it remains in $W \subset I_\infty$). A computational scheme then proceeds by implementing steps 3, 4 and 5 given above. Note that in this situation, no requirement of a stable plant is necessary.

7 Examples

Two examples are presented which demonstrate the computational schemes described previously.

7.1 Example 1: An open-loop stable system

Consider the following stable dynamics taken from [10]:

$$x(k+1) = \begin{pmatrix} 4/3 & -2/3 \\ 1 & 0 \end{pmatrix} x(k) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(k) \tag{18}$$

subject to the saturation constraint:

$$|u| \leq \psi$$

We choose:

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R = 1 \tag{19}$$

and the set of initial conditions is taken to be:

$$I = \{(x_1, x_2) : |x_1| \leq 2, |x_2| \leq 2\} \tag{20}$$

Three different levels of control saturation are considered: $\psi = 1$, $\psi = 0.5$, and $\psi = 0.1$.

Following the procedure outlined in the previous section, the relevant parameters are tabulated in Tables 1, 2 and 3. Specifically, α_N, ρ_N and ζ_N were calculated using the nonlinear programming software NPSOL according to the optimization problems setup in step 3 of the previous section. For reference, parameters for the unconstrained system (uncon.) are also tabulated.

Recall that either $\gamma_N < 1$ or $\zeta_N < 1$ determines a sufficient condition for stability. As expected, as the constraint is tightened, both stability parameters γ_N and ζ_N increase (Table 1). Considering the results using γ_N, when no constraint is present, a horizon of 2 is sufficient for guaranteed stability. As the constraint is tightened from $\psi = 1$ to $\psi = 0.1$, the sufficient horizon length shifts from 4 to 5.

As in the unconstrained case, the parameter ζ_N is a less conservative estimate of stability than γ_N. In the unconstrained case, ζ_N predicts a horizon of only $N = 1$ to guarantee stability. Furthermore, we find that for all three levels of saturation ($\psi = 1,\ 0.5,\ 0.1$) ζ_N is less that 1 for $N = 3$, implying stability.

	Stability: γ_N				Stability: ζ_N			
N	uncon.	$\lvert u\rvert \leq 1$	$\lvert u\rvert \leq .5$	$\lvert u\rvert \leq .1$	uncon.	$\lvert u\rvert \leq 1$	$\lvert u\rvert \leq .5$	$\lvert u\rvert \leq .1$
1	2.0000	2.8082	2.9383	3.0490	<u>0.6862</u>	1.6604	1.7588	1.8387
2	<u>0.7990</u>	1.7054	1.8080	1.8928	0.4678	1.1222	1.1925	1.2526
3	0.7165	1.1710	1.2601	1.3411	0.4683	<u>0.9224</u>	<u>0.9581</u>	<u>0.9925</u>
4	0.7109	<u>0.9457</u>	1.0059	1.0802	0.4641	0.8954	0.9123	0.9237
5	0.7077	0.9025	<u>0.9212</u>	<u>0.9707</u>	0.4634	0.8999	0.9221	0.9403
6	0.7073	0.9035	0.9277	0.9508	0.4634	0.8961	0.9253	0.9541
7	0.7072	0.8972	0.9256	0.9543	0.4634	0.8914	0.9152	0.9438
8	0.7072	0.8924	0.9159	0.9464	0.4634	0.8911	0.9087	0.9281
9	0.7072	0.8922	0.9091	0.9327	0.4634	0.8909	0.9084	0.9226
10	0.7072	0.8920	0.9086	0.9243	0.4634	0.8909	0.9081	0.9237

Table 1: Example 1: Stability parameters γ_N and ζ_N

The performance bound $\mathcal{P}_N$, given by[1]:

$$\mathcal{P}_N^{\gamma} = \left(1 + \left(\frac{\alpha_{N-1} - 1}{\alpha_{N-1}}\right) \frac{\gamma_N}{1 - \gamma_N}\right)$$

which bounds the infinite horizon cost of the RH policy ($J_{\hat{u}_N}$) from the optimal infinite horizon cost (J) by:

$$J_N(x) \leq J(x) \leq J_{\hat{u}_N}(x) \leq \mathcal{P} J_N(x)$$

[1]When using ζ_N we denote it by $\mathcal{P}_N^{\zeta} = \left(1 + \left(\frac{\alpha_{N-1}-1}{\alpha_{N-1}}\right) \frac{\zeta_N}{1-\zeta_N}\right)$

	Performance: $\mathcal{P}_N^\gamma$				Performance: $\mathcal{P}_N^\zeta$			
N	uncon.	$\lvert u\rvert \leq 1$	$\lvert u\rvert \leq .5$	$\lvert u\rvert \leq .1$	uncon.	$\lvert u\rvert \leq 1$	$\lvert u\rvert \leq .5$	$\lvert u\rvert \leq .1$
1	—	—	—	—	2.4581	—	—	—
2	1.4824	—	—	—	1.1067	—	—	—
3	1.0390	—	—	—	1.0136	3.9000	7.5798	43.9068
4	1.0132	2.0392	—	—	1.0047	1.5111	2.0670	2.8762
5	1.0016	1.1288	1.2226	2.9469	1.0006	1.1250	1.2253	1.9251
6	1.0001	1.1274	1.3079	1.7092	1.0000	1.1174	1.2974	1.7629
7	1.0000	1.0520	1.2464	1.7720	1.0000	1.0490	1.2137	1.6210
8	1.0000	1.0050	1.0939	1.4744	1.0000	1.0049	1.0858	1.3467
9	1.0000	1.0025	1.0110	1.1672	1.0000	1.0024	1.0109	1.1437
10	1.0000	1.0008	1.0060	1.0353	1.0000	1.0008	1.0059	1.0350

Table 2: Example 1: Performance bounds $\mathcal{P}_N^\gamma$ and $\mathcal{P}_N^\zeta$ (using γ_N and ζ_N)

(cf. Thm. 5.4) is presented in Table 2

As the constraint is tightened, a clear degradation of the performance bound ($\mathcal{P}_N$) is revealed. Under γ_N, for the unconstrained problem, a horizon length of 3 is sufficient to guarantee performance within 5% of the optimal. This same level of guaranteed performance requires horizon lengths of 8, 9 and 10 for $\psi = 1$, $\psi = 0.5$, and $\psi = 0.1$ respectively.

Similar results are obtained using ζ_N. For the unconstrained problem, a horizon length of 3 is sufficient to guarantee a performance within 5% of the optimal. This same level of guaranteed performance requires horizon lengths of 7, 9 and 10 for $\psi = 1$, $\psi = 0.5$, and $\psi = 0.1$ respectively.

The parameters α_N and ρ_N are displayed in Table 3 for reference. These parameters are necessary to calculate γ_N and additionally, α_N is needed in the performance bound $\mathcal{P}_N$ in conjunction with both γ_N and ζ_N. It is interesting to note that α_N actually does not decrease monotonically in N.

	α_{N-1}			ρ_N		
N	$\lvert u\rvert \leq 1$	$\lvert u\rvert \leq .5$	$\lvert u\rvert \leq .1$	$\lvert u\rvert \leq 1$	$\lvert u\rvert \leq .5$	$\lvert u\rvert \leq .1$
1	3.8082	3.9382	4.0491	0.2626	0.2539	0.2470
2	1.9872	2.0822	2.1610	0.1418	0.1317	0.1241
3	1.3227	1.4040	1.4798	0.1147	0.1025	0.0937
4	1.0635	1.1143	1.1834	0.1108	0.0973	0.0872
5	1.0141	1.0194	1.0624	0.1100	0.0963	0.0863
6	1.0138	1.0246	1.0381	0.1088	0.0946	0.0841
7	1.0060	1.0202	1.0384	0.1082	0.0927	0.0810
8	1.0006	1.0087	1.0276	0.1081	0.0920	0.0790
9	1.0003	1.0011	1.0122	0.1081	0.0919	0.0785
10	1.0001	1.0006	1.0029	0.1081	0.0919	0.0784

Table 3: Example 1: Parameters α_N and ρ_N (used to calculate γ_N)

N	Stability: γ_N			Stability: ζ_N		
	uncon.	output	mixed	uncon.	output	mixed
1	7.3815	7.3815	7.3815	1.2799	1.2799	1.2799
2	1.7319	1.7319	1.7319	<u>0.7746</u>	<u>0.7746</u>	<u>0.8470</u>
3	1.1808	1.1808	1.2761	0.6581	0.6804	0.8433
4	<u>0.9668</u>	1.0034	1.1147	0.6415	0.6446	0.7702
5	0.9230	<u>0.9545</u>	<u>0.9831</u>	0.6420	0.6420	0.7352
6	0.9195	0.9278	0.9329	0.6420	0.6420	0.7251
7	0.9194	0.9197	0.9199	0.6419	0.6419	0.7246
8	0.9191	0.9193	0.9193	0.6419	0.6419	0.7246
9	0.9191	0.9192	0.9192	0.6419	0.6419	0.7246
10	0.9191	0.9191	0.9191	0.6419	0.6419	0.7246

Table 4: Example 2: Stability parameters γ_N and ζ_N

7.2 Example 2: An open-loop unstable system

Consider the unstable system

$$x(k+1) = \begin{pmatrix} 1 & 1.1 \\ -1.1 & 1 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(k)$$

$$y(k) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} x(k)$$

We impose the following output constraints:

$$|y_1(k)| \leq 2, \quad |y_2(k)| \leq 0.5 \tag{21}$$

These constraints bound a feasible region (subset of I_∞) in state space and allow for calculations of α_N and ρ_N over this region. Also, we take

$$Q = C^T C = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}, \qquad R = 1$$

Calculations were repeated with the addition of the following mixed constraint:

$$x_1 - u \leq 0.3. \tag{22}$$

Results are presented in Tables 4, 5 and 6. When only the output constraints (21) are imposed, this is denoted "output". The columns headed "mixed" correspond to the addition of the mixed constraint (22).

Table 4 presents the stability parameters γ_N and ζ_N. In the unconstrained case, using γ_N, stability is guaranteed ($\gamma_N < 1$) at a horizon length of $N = 4$, while in both of the constrained cases, a horizon length of $N = 5$ is required. The less conservative estimates given by ζ_N indicate that a horizon of only $N = 2$ is needed in all three cases.

Performance results (Table 5) under γ_N indicate that a performance within 1% of optimal is assured for a horizon length of 6 for the unconstrained problem, 7 for output constraints only, and 8 when the mixed constraint is included. Using ζ, a 1% level of performance is obtained for $N = 5$ in the unconstrained case, and $N = 7$ for both of the constrained cases.

	Performance: $\mathcal{P}_N^\gamma$			Performance: $\mathcal{P}_N^\zeta$		
N	unconstrained	output	mixed	unconstrained	output	mixed
1	—	—	—	—	—	—
2	—	—	—	2.6242	2.6242	3.6157
3	—	—	—	1.4273	1.4726	2.5072
4	2.4411	—	—	1.0885	1.1525	1.5885
5	1.0527	1.7792	4.7800	1.0079	1.0666	1.1809
6	1.0060	1.1222	1.2067	1.0009	1.0171	1.0392
7	1.0038	1.0080	1.0103	1.0006	1.0013	1.0024
8	1.0012	1.0023	1.0023	1.0002	1.0004	1.0005
9	1.0002	1.0011	1.0011	1.0000	1.0002	1.0003
10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 5: Example 2: Performance bounds $\mathcal{P}_N^\gamma$ and $\mathcal{P}_N^\zeta$ (using γ_N and ζ_N)

	α_{N-1}			ρ_N		
N	unconstrained	output	mixed	unconstrained	output	mixed
1	8.3814	8.3814	8.3814	0.1193	0.1193	0.1193
2	1.8957	1.8957	1.8957	0.0864	0.0864	0.0864
3	1.2853	1.2853	1.3890	0.0813	0.0813	0.0813
4	1.0521	1.0918	1.2130	0.0810	0.0810	0.0810
5	1.0044	1.0386	1.0697	0.0810	0.0810	0.0810
6	1.0005	1.0096	1.0151	0.0810	0.0810	0.0810
7	1.0003	1.0007	1.0009	0.0809	0.0809	0.0809
8	1.0001	1.0002	1.0002	0.0809	0.0809	0.0809
9	1.0000	1.0001	1.0001	0.0809	0.0809	0.0809
10	1.0000	1.0000	1.0000	0.0809	0.0809	0.0809

Table 6: Example 2: Stability parameters α_N and ρ_N (used to calculate γ_N)

Once again, the parameters α_N and ρ_N are included for reference in Table 6. In this case, we run across the interesting fact that ρ_N is not affected by the constraints. This is due to ρ_N being the *minimum* of $x^TQx/J_N(x)$. If this minimum always occurs when the constraints are not active, then ρ_N will not be affected by the constraints.

8 Concluding Remarks

We presented results concerning the feasibility, stability and performance of constrained finite receding horizon linear quadratic control without using end constraints. It was proven that a sufficiently long horizon will guarantee feasibility and stability with the finite horizon cost J_N being a Lyapunov function. In this case, bounds on the infinite horizon cost of the finite receding horizon controller were derived. In addition, it was shown that off–line finite horizon computations are possible for certain classes of systems, which guarantee stability and performance bounds for the constrained RH policy. Generalizations of all of the results in this paper to constrained nonlinear optimal control are possible. These results will be submitted for publication shortly.

A Review of relevant concepts from real analysis

Definition A.1 *A set K is* compact *if every open covering of K has a finite subcovering.*

In $\mathbb{R}^n$, a set is compact iff it is closed and bounded.

Definition A.2 *A function φ is called* upper semi-continuous *if for each real number α, the set $\{x : \varphi(x) < \alpha\}$ is open.*

For our purposes, an equivalent definition of upper semi-continuity can be given by:

Definition A.3 *A function φ is* upper semi-continuous *at y if $\varphi(y) \neq +\infty$ and*

$$\varphi(y) \geq \overline{\lim}_{x \to y} \varphi(x)$$

where

$$\overline{\lim_{x \to y}} \varphi(x) = \inf_{\delta > 0} \sup_{0 < |x-y| < \delta} \varphi(x).$$

A function is upper semi-continuous on a set if it is upper semi-continuous at every point of the set.

Theorem A.1 *Let φ be an upper semi-continuous function on a compact space X. Then φ is bounded from above and assumes its maximum.*

Proof [13] ∎

Theorem A.2 *(Dini): Let φ_n be a sequence of upper semi-continuous real-valued functions on a compact space X, and suppose that for each $x \in X$ the sequence $\varphi_n(x)$ decreases monotonically to zero. Then φ_n converges to zero uniformly.*

Proof [13]: Choose $\epsilon > 0$, and let $O_n = \{x : \varphi_n(x) < \epsilon\}$. Since φ_n is upper semi-continuous, O_n is open. Since $\varphi_n(x) \to 0$ for each x, we have $X \subset \bigcup O_n$. By the compactness of X, there are a finite number of open sets $\{O_1, ..., O_n\}$ whose union contains X. But this implies that $O_n = X$, and hence $\varphi_N(x) < \epsilon$ for all x. If $n \geq N$, we have $0 \leq \varphi_n(x) \leq \varphi_N(x) < \epsilon$, and the sequence $\{\varphi_n\}$ converges to 0 uniformly. ∎

B Arbitrary Terminal Weight P_0

In proving results for $P_0 = Q$, we used extensively that the finite horizon costs J_N increase monotonically. For arbitrary P_0, we do not have this property. Yet, we can bound the costs that correspond to an arbitrary P_0 from below by the cost with $P_0 = Q$ and from above by the cost with $P_0 = \infty$ (end constraints). Below, we briefly show how simple modifications to the results for $P_0 = Q$ allow for arbitrary terminal weights.

Let's establish the following notation: Let J_N^0 correspond to the terminal weight Q, and J_N^∞ corresponding to end constraints $P_0 = \infty$. For arbitrary P_0 we will use J_N.

Lemma B.1 *Let $P_0 = \infty$ (i.e. end constraints). Let $\mu > 0$ and consider the set S_μ, then:*

1. $J_N^\infty(x) \to J(x)$ *uniformly*

2. Define $\varphi_N^\infty(x) = \frac{J_N^\infty(x) - J(x)}{x^T Q x}$, $\forall x \neq 0$, *and* $\varphi_N(0) = \overline{\lim}_{x \to 0} \varphi_N(x)$. *Then* $\varphi_N^\infty \to 0$ *uniformly.*

Proof This is a simple consequence of Dini's Theorem since the J_N^∞ are a sequence of decreasing functions (See [5] for the fact that $J_N^\infty \to J$). The only thing to show is that the end constraint is uniformly feasible for every initial condition in S_μ. This follows by arguments similar to those in the proof of Lemma 4.3. For some N large enough the state must enter a neighborhood of the origin (cf. Lemma 4.3) from which deadbeat control may be performed. ∎

We note that since $J_N^0 \leq J_N \leq J_N^\infty$, then the above lemma also holds for arbitrary P_0.

Feasibility
The feasibility theorem holds as stated, with only a minor change needed for its proof.

Theorem B.1 *Let* $\mu > 0$ *be fixed and consider the* μ *sub-level set of* $J(x)$, *(i.e.* S_μ*). Then there exists an* N' *such that for any* $n \geq N'$, S_μ *is RH n-invariant.*

Proof The bulk of the proof follows exactly the same as the proof of Theorem 4.2 for $P_0 = Q$. The only difference comes in the final chain of inequalities used to construct the contradiction. In this case, first we note that since $J_N \to J$ on S_μ, then we can choose N large enough so that $\mu + \frac{\epsilon}{4} \geq J_N(x)$ for any $x \in S_\mu$ and $J_{N-1}^0(x(1)) \geq \mu - \frac{\epsilon}{2}$. Hence we may obtain the final contradiction paralleling equation (8) as follows:

$$
\begin{aligned}
\mu + \frac{\epsilon}{4} &\geq J_N(x(0)) \\
&= x^T(0)Qx(0) + u^T R u + J_{N-1}(x(1)) \\
&\geq x^T(0)Qx(0) + u^T R u + J_{N-1}^0(x(1)) \\
&\geq \epsilon + \mu - \frac{\epsilon}{2} \\
&= \mu + \frac{\epsilon}{2}
\end{aligned}
$$

Hence showing that the theorem is valid for arbitrary P_0. ∎

Stability
The stability theorem holds as well for arbitrary P_0.

Theorem B.2 *Let* I *be a compact subset of* I_∞. *Then there exists an* N^* *such that for* $N \geq N^*$, *the receding horizon policy is stabilizing for any initial condition in* I, *and* J_{N^*} *serves as a Lyapunov function.*

Proof Once again, the proof relies on the uniform convergence of the functions φ_N, which is guaranteed by the Lemma given above. Hence, the same proof (Theorem 5.2) follows without major change. ∎

Performance
The performance result now also holds with a minor change to account for the fact that α may be less than 1.

Theorem B.3 *Let N and γ_N be as in Theorem B.2. Denote the infinite horizon performance using the receding horizon policy $u(k) = \hat{u}_N(x(k))$ by:*

$$J_{\hat{u}_N}(x(0)) = \sum_{k=0}^{\infty} x^T(k)Qx(k) + u^T(k)Ru(k) \tag{23}$$

Then a bound for the infinite-horizon performance is given by:

$$J(x(0)) \leq J_{\hat{u}_N}(x(0)) \leq \mathcal{P}_N J_N(x(0)). \tag{24}$$

where

$$\mathcal{P}_N = \left(1 + \left(\frac{max\{0, \alpha_{N-1} - 1\}}{\alpha_{N-1}}\right) \frac{\gamma_N}{1 - \gamma_N}\right) \tag{25}$$

Proof The proof of this is almost identical to that for Theorem 5.4, and additionally can be found in [9]. ∎

Note that for $\alpha \leq 1$, the performance bound is given exactly by the finite horizon cost J_N.

References

[1] D.P. Bersekas, "Dynamic Programming and Stochastic Control", *Academic Press* New York, 1976.

[2] R.R. Bitmead, M. Gevers, I.R. Petersen, and R.J. Kaye, "Monotonicity and stabilizability properties of solutions of the Riccati difference equation: propositions, lemmas, theorems, fallacious conjectures and counterexamples", In *Systems and Control Letters*, 5:309–315, 1985.

[3] R.R. Bitmead, M. Gevers, and V. Wertz, "Optimal control redesign of generalized predictive control", In *Proc. IFAC Symp. on Adaptive Systems in Control and Signal Proc.*, 129–134, 1989

[4] C.R. Cutler and B.L. Ramaker, "Dynamic matrix control–a computer algorithm", In *AIChE National Meeting, Houston, TX*, Apr. 1979.

[5] S.S. Keerthi and E.G. Gilbert, "Optimal Infinite-Horizon Feedback Laws for a General Class of Constrained Discrete-Time Systems: Stability and Moving-Horizon Approximations", In *Journal of Opt. Theory and Appl.*, Vol 57., No. 2, May 1988.

[6] H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems*, Wiley-Interscience, New York, 1972.

[7] W.H. Kwon and A.E. Pearson, "On feedback stabilization of time-varying discrete linear systems", In *IEEE Trans. Auto. Control*, AC–23:479–481, 1978.

[8] W.H. Kwon and A.E. Pearson, "A modified quadratic cost problem and feedback stabilization of a linear system", In *IEEE Trans. Auto. Control*, AC–22:838–842, 1977.

[9] V. Nevistić and J. Primbs, "Finite Receding Horizon Linear Quadratic Control: A Unifying Theory for Stability and Performance Analysis", California Institute of Technology, CDS Technical Memo, CIT-CDS 97-001, Jan. 1997.

[10] J.B. Rawlings and K.R. Muske, "The stability of constrained receding horizon control", In *IEEE Trans. Auto. Control*, Vol 38, No. 10:1512–1516, 1993.

[11] J.S. Shamma and D. Xiong, "Linear Non-Quadratic Optimal Control", To appear in *IEEE Trans. Auto. Control*, 1997.

[12] J. Richalet, A. Rault, J.L. Testud, and J. Papon, "Model predictive heuristic control: Applications to industrial processes", In *Automatica*, vol. 14, pp. 413–428, 1978.

[13] H.L. Royden, *Real Analysis, third edition*, Macmillan, New York, 1988.